Unit 2

Modeling Motion

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Each day you confront motion in nearly everything you do. You tie your shoes, ride or walk to school, open and close books, and write notes. You see aircraft fly overhead and you see the sun rise in the east, move across the sky, and set in the west. You might throw, kick, or hit a ball. Not all of these motions are easily modeled mathematically. However, motion along a line and motion along a circle are easily modeled. In this unit, you will investigate an important tool for modeling motion—vectors. Vectors are especially useful for modeling motions because they can represent distance and direction, important descriptors of motions, simultaneously.

Planned, as well as actual, routes of boats and ships involve linear motion. Think about how you might describe or represent a planned route on a map. Think also about conditions that might affect a planned route and how you might incorporate that information in the planning process.

Think About This Situation

Suppose you wanted to map a route that involved sailing 3 km west from Bayview Harbor to Presque Island, then 6 km south to Rudy Point, and then 5 km southeast to Traverse Bay.

a. How could you represent the planned route geometrically?

b. How could you represent a sailing trip directly from Bayview Harbor to Traverse Bay?

c. How could you estimate the length of the route in Part b? How would you describe its direction?

d. How would a northwest water current affect the path you would steer the boat to maintain the route in Part b?
INVESTIGATION 1 Navigation: What Direction and How Far?

Vectors and vector operations are used extensively in navigation on water and in the air. Imagine that you are navigating a boat along the small portion of the Massachusetts coast shown in the nautical chart below. Note that within the chart itself there are several aids to navigation such as mileage, scales, landmarks, and buoys. The buoys are painted red or green and may have a red or green flashing light.

1. As a class, begin by examining some of the information provided by the chart.
   a. Some of the buoys are represented by two concentric circles and a diamond. Which of these have noise-making devices? What are they?
   b. Which buoys have flashing lights? What is each color? How often does the light flash?
   c. What symbol is used to represent easily recognized landmarks on the chart? What landmarks are shown?
   d. What do you think the dotted lines on the chart represent? Why is this important knowledge for navigation?
e. At the right of the chart is a nautical mile (nm) scale. Use this scale to find the distance from the “SH” buoy to the “GP” buoy. Measure from circle center to circle center on the chart.

f. There are other scales at the top and along the right edge of the chart. What do you think these scales represent?

g. What other scale on this chart can be used to measure nautical miles? What does a nautical mile represent based on this scale?

h. A nautical mile is 6,076.1033 feet. How does a nautical mile compare to a statute mile (regular mile)?

Coastal water nautical charts are designed so that the top is due north and the right side is due east. The heading of a boat is given in degrees clockwise from due north. Thus, due north is 0˚, due east is 90˚, due south is 180˚, and due west is 270˚. A 60˚ heading through a channel northwest of Anguilla Island is depicted below.

2. Use a copy of the nautical chart on page 81 to complete this activity. Measure distances to the nearest \( \frac{1}{10} \) nm, using a ruler made from the nautical mile scale. Measure angles to the nearest degree using a protractor.

a. Mark and label a point \( P \) on the chart to represent a boat that is 3 nautical miles from the “3” bell on a heading of 200˚. What buoy is nearest to \( P \)?

b. Draw an arrow from the “SH” buoy to the “6” buoy. What is the heading? What is the distance in nautical miles?

c. What are the heading and distance of the path from the “6” buoy to the center of the mouth of the channel at Stone Harbor?

d. A public launching ramp is located on the channel near the Launch Center. Draw an arrow showing a route from the easterly end of the ramp to the “SH” buoy. Find the heading and distance to the “SH” buoy.

e. Why are arrows particularly useful representations for nautical paths?
3. The arrows showing boating routes are directed line segments. They have both a magnitude (length) and a direction (heading). Thus, an arrow is a geometric representation of a vector—a quantity with magnitude and direction.

a. Accurately draw arrows representing vectors with the following characteristics.
   - Length: 5 cm; heading: 100°
   - Magnitude: 7 cm; heading: 293°
   - Length: 2 nm; heading: 87° (Use the chart scale for nautical miles.)

b. Draw an arrow for each vector described.
   - A boat with speed of 2 knots (nautical miles per hour) on a 142° heading
   - A force of 5 pounds on a heading of 90°
   - A speed of 60 mph on a heading of 270°
   - A force of a 15 mph wind blowing from a heading of 300°

c. Compare the arrows you drew in Parts a and b with those of a classmate. Resolve any differences.

d. Since arrows representing vectors can be drawn anywhere, as in Parts a and b, it is important to know whether two arrows drawn using the same scale represent the same or different vectors. Explain why the following method provides a geometric test for the equality of the vectors represented by arrows \( \overrightarrow{PQ} \) and \( \overrightarrow{RS} \).

   Step 1  Connect the heads \( Q \) and \( S \) and connect the tails \( P \) and \( R \).

   Step 2  If \( PQSR \) is a parallelogram, then vector \( \overrightarrow{PQ} = \overrightarrow{RS} \).

In the activities that follow, when the instructions ask you to make an “accurate drawing” or an “accurate sketch” of a vector, you should draw an arrow using a straight edge and use a ruler and protractor to measure. If, however, the instructions are simply “sketch” or “draw” a vector, you should make a freehand sketch of an arrow that approximates the characteristics of an accurately drawn vector. Use the freehand sketch to guide your thinking, and use an accurate drawing when you need accurate estimates for angle measures and segment lengths. Note that “draw a vector” actually means “draw a geometric representation of the vector” (an arrow).
4. Suppose a fishing boat leaves the Stone Harbor channel on a heading of 25° at a speed of 1.5 knots (nautical miles per hour).

a. On your copy of the nautical chart, sketch the vector representing the distance and direction traveled during the first hour of the trip. Describe the boat’s location after one hour.

b. Describe how you could use the vector in Part a to determine the vector for a 3-hour trip at the same speed and heading. Sketch this vector. Describe how you could locate the fishing boat at the end of 1.5 hours, 2 hours, and 2.75 hours.

c. On a piece of plain or graph paper, draw a vector about 10 cm long. Sketch a vector that is half this vector. Now sketch another vector whose length is half that of the original vector. Are the sketched vectors equal? Must they be equal? Explain.

d. In general, how would you sketch a vector that was \( n \) times a given vector? How are the lengths and headings of these two vectors related?

5. Now suppose a boat begins a trip at the mouth of the channel at Stone Harbor at a heading of 20° and a speed of 2 knots.

a. Sketch the vector showing the position at the end of the first hour.

b. Suppose the boat returns to the harbor along the same route at the same speed. Sketch the return vector and give its magnitude and heading.

c. The word “opposites” can be used to denote the vectors in Parts a and b. How is this word descriptive of the relationship?

d. Sketch a vector opposite to the vector in Part a from the “3” bell. Give its magnitude and heading. Compare your results to those in Part b.
Vectors are quantities with magnitude and direction that are represented geometrically by arrows.

**a.** Describe how you know when two arrows represent the same vector.

**b.** How are a vector and a multiple of that vector similar? How are they different?

**c.** What do you think is always true about the magnitudes and directions of any two opposite vectors?

*Be prepared to explain your group’s thinking to the entire class.*

Vectors can be denoted in various ways. One way is to use italicized letters with arrow shapes over them, such as \( \vec{a} \) or \( \vec{v} \). When the initial point or tail and terminal point or head are labeled, then capitalized, italicized letters such as \( \overrightarrow{AB} \) can be used. Since a vector \( \vec{v} \) is determined by its magnitude \( r \), and its heading \( \theta \), we could also represent \( \vec{v} \) symbolically as \([r, \theta]\). \( \theta \) is the Greek letter “theta.” Often, Greek letters are used to denote measures of angles.

![Diagram of vector notation](image)

When a vector \( \vec{a} \) is multiplied by a real number \( n \), the number is called a **scalar** and the product, \( n\vec{a} \), is a **scalar multiple** of \( \vec{a} \). (In a similar manner, \( n\overrightarrow{AB} \) is a scalar multiple of the vector \( \overrightarrow{AB} \).) When \( n > 0 \), \( n\vec{a} \) is the vector whose length is \( n \) times the length of \( \vec{a} \) and has the same direction as \( \vec{a} \) as pictured below. When \( n < 0 \), the length of \( n\vec{a} \) is \(|n| \) times the length of \( \vec{a} \); but \( n\vec{a} \) points in the opposite direction. The opposite of a vector \( \vec{a} \) or \( \overrightarrow{AB} \) is denoted \(-\vec{a}\) or \(-\overrightarrow{AB}\). If \( \vec{a} = [r, \theta] \), then \( n\vec{a} = [nr, \theta] \) when \( n > 0 \), and \([|n| r, \theta + 180^\circ] \) when \( n < 0 \); its opposite is \([r, \theta + 180^\circ] \).
On Your Own

On a copy of this map of Lake Michigan, plot the courses described below.

![Map of Lake Michigan with cities marked]

a. Daily ferries shuttle people and cars between Manitowoc, Wisconsin, and Ludington, Michigan. Draw the vector for the ferry route from Manitowoc to Ludington. Label it \( \vec{v} \). Find its magnitude and heading.

b. Find the magnitude and heading of \(-\vec{v}\) and describe what it represents. Draw \(-\vec{v}\) beginning at Charlevoix, Michigan.

c. Draw the vector for the direct course to sail from South Haven, Michigan, to Milwaukee, Wisconsin. Call it \( \vec{a} \). Find the magnitude and heading of \( \vec{a} \).

d. Sketch 0.5\( \vec{a} \) from Grand Haven, Michigan, and find its magnitude and heading.
LESSON 1 • MODELING LINEAR MOTION

INVESTIGATION 2 Changing Course

In the previous investigation, you used vectors to model straight-line paths. In this investigation, you will explore how vectors can be used to model routes when there is a change of course during the trip. For Activity 1, you will need a copy of the nautical chart from Investigation 1.

1. Suppose Natalie, the skipper of the fishing boat High Hopes, left the mouth of the Stone Harbor channel making 6 knots at 65°. She traveled for 20 minutes, then turned to a heading of 350° and traveled for 5 minutes before deciding to drop anchor and begin fishing.

   a. Using a copy of the nautical chart from Investigation 1, draw an accurate vector diagram showing the paths taken and the position of the High Hopes at the end of 25 minutes. (Save this chart for later use.) What units measure the lengths of these vectors? How long are these vectors?

   b. Suppose the fish are biting and Natalie wants to inform Keith, the skipper of the Little Hope, where she is located so that he can join her. Accurately draw a vector representing the path Keith should take from the mouth of the channel directly to the High Hopes. What heading should she advise him to take? How far will he need to travel?

   c. Suppose Natalie did not drop anchor until she traveled with a heading of 350° for 30 minutes instead of 5 minutes. In this case, what course should she advise Keith to take from the mouth of the channel?

   d. The vector representing the path that Keith should travel to the good fishing spot is called the sum or resultant of the two vectors describing the route taken by the High Hopes. How are the initial and terminal points of the resultant vector in Parts b and c related to the two vectors representing the trips taken by the High Hopes?

   e. Suppose Natalie had left the harbor at a speed of 6 knots on a heading of 350° for 30 minutes and then turned to a heading of 65° and traveled for 20 minutes. Draw an accurate vector diagram of Natalie’s path and the resultant vector. Describe the resultant vector in terms of heading and magnitude. How does this resultant vector compare to the resultant vector found in Part c?
2. Consider the following vectors: \( \vec{a} \) (magnitude 5 cm, heading 20°), \( \vec{b} \) (magnitude 4 cm, heading 60°), \( \vec{c} \) (magnitude 4 cm, heading 100°), and \( \vec{d} \) (magnitude 3 cm, heading 200°). Make accurate sketches of each vector sum and measure to find the magnitude (to the nearest 0.1 cm) and heading (to the nearest 5°) for each resultant vector.
   a. \( \vec{a} + \vec{b} \)
   b. \( \vec{a} + \vec{d} \)
   c. \( \vec{a} + \vec{b} + \vec{c} \)
   d. \( \vec{a} + \vec{b} + \vec{c} + \vec{d} \)

3. Now investigate some general properties of vector addition. Begin by sketching any two vectors \( \vec{a} \) and \( \vec{b} \).
   a. Draw diagrams showing how to find \( \vec{a} + \vec{b} \) and \( \vec{b} + \vec{a} \). What do you notice? Compare your observations to those of another group.
   b. To which property of real number operations is this similar?
   c. Choose a point in the plane. Starting at the chosen point, draw a single diagram showing how to find \( \vec{a} + \vec{b} \) and \( \vec{b} + \vec{a} \). What shape is formed? Prove your conjecture.

4. On a sheet of plain or graph paper, make an accurate drawing of a vector \( \vec{a} \) with magnitude 4 cm and heading 253° and a vector \( \vec{v} \) with magnitude 5 cm and heading 22°.
   a. Without measuring, find the magnitude and heading of as many of the following vectors as possible. Explain your reasoning in each case.
      i. \( 2 \cdot \vec{a} \)
      ii. \( \frac{1}{2} \cdot \vec{v} \)
      iii. \( \vec{v} + \vec{a} \)
      iv. \( \vec{a} + \vec{v} \)
      v. \( 3(\vec{a} + \vec{v}) \)
      vi. \( 3\vec{a} + 3\vec{v} \)
      vii. \( -2\vec{v} \)
      viii. \( 2\vec{v} + (-2\vec{a}) \)
      ix. \( 2\vec{v} + 3\vec{a} \)
      x. \( -2\vec{v} + (-2\vec{a}) \)
   b. Find the magnitude and heading of the remaining vectors by measuring. Use as few drawings as possible. Look for possible connections between pairs of vectors that might reduce your work.
   c. What general rule is suggested by Parts v and vi? Test your conjecture.

5. Retrieve your drawing for Activity 1 Part c.
   a. Sketch three additional two-leg routes to the fishing grounds. In each case, find the resultant vector.
   b. How are the resultant vectors of Part a related to the resultant vector of Activity 1 Part c?
   c. Decide if each of the following conjectures is true or false. In each case, explain your reasoning.
      ■ The vector sum of any two given vectors is unique.
      ■ If a vector is the sum of two given vectors, it cannot be the sum of two different vectors.
6. On the chart below are drawn a vector with heading 90° and a vector with heading 0° that give one route to the good fishing area identified in Activity 1.

![Vector Chart](image)

**a.** Measure to find the magnitudes of these two vectors. Then use those measures to calculate (not measure) the magnitude of the resultant.

**b.** Starting at the harbor, is it possible to find another pair of vectors with direction 0° and 90° that have the vector sum in Part a? Explain your reasoning.

**c.** Recall that for a right \( \triangle ABC \):

\[
\sin A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}
\]

\[
\cos A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}
\]

\[
\tan A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}
\]

Use a pair of perpendicular vectors to compute the heading to the good fishing area. Compare your computed heading with that obtained by measuring in Activity 1 Part b.
7. Now investigate further how a vector can be thought of in terms of the sum of horizontal and vertical vectors called its **components**.

a. Suppose a vector represents a 2-nautical mile route with a heading of 12°. Compute the lengths of the north (0°) and east (90°) legs of a route to the same location.

b. Suppose a vector \( \vec{v} \) represents a 2 nm route with a heading of 325°. Make a sketch of the vector \( \vec{v} \) and include the north and west vectors that would give the resultant vector \( \vec{v} \).

- What are the measures of the angles of the triangle formed by these three vectors?
- Compute the magnitudes of the north and west vectors.
- What are the headings for the north and west vectors?

c. Compute the magnitudes of the horizontal and vertical components of a 2 nm vector with a heading of 120°. What are the directions of the components?

d. How would you compute the magnitudes of the horizontal and vertical components of any 2 nm vector with a heading between 180° and 270°? Any 5 nm vector with a heading between 180° and 270°? Compare your methods with those of other groups and resolve any differences.

**Checkpoint**

In this investigation, you explored the geometry of the addition of vectors.

- Describe what is meant by the resultant or sum of two vectors.

- Any nonzero vector can be represented as the sum of a horizontal vector and a vertical vector. Illustrate and explain how this can be done for a vector whose direction is given as a heading.

- In the vector diagram below, \( \overrightarrow{AC} \) and \( \overrightarrow{CB} \) are the horizontal and vertical components of \( \overrightarrow{AB} \) respectively.

  - If you know the heading and magnitude of \( \overrightarrow{AB} \), how would you calculate the magnitudes of \( \overrightarrow{AC} \) and \( \overrightarrow{CB} \)?
  - If you know the magnitudes of \( \overrightarrow{AC} \) and \( \overrightarrow{CB} \), how would you calculate the magnitude and heading of \( \overrightarrow{AB} \)?

*Be prepared to share your descriptions and illustrations with the class.*

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**UNIT 2 • MODELING MOTION**
On Your Own

Refer to the nautical chart used in the investigation. Suppose Keith is fishing off Great Point due west of the cupola and due north of the “GP” buoy when he receives a report of good fishing due south of the stack and due west of the “3” buoy.

a. What heading should Keith set to get to the good fishing spot? How far will he need to travel?

b. Make an accurate drawing of the components of the route in Part a. Using scale measurements, find the length, in nautical miles, of the components.

c. Use the vector right triangle you have drawn to calculate the lengths of the components in nautical miles. Compare the calculated and measured lengths.

INVESTIGATION 3 Go with the Flow

The vector models you have been using for navigation assume that the force moving a boat is the only one acting on the craft. When this is the case, the craft moves in a straight line in the direction of the force. However, what happens when two (or more) forces act simultaneously on an object? For example, tides in the ocean are forces on boats that move the boats in the direction of the tide. Sailing ships without motors use tidal flows to help them enter and leave port. The wind, too, is a force that affects the path a boat or an airplane follows. A fundamental principle of physics is that the effect of two forces acting on a body is the sum of the forces. In this investigation, you will learn how to use this principle.

1. Suppose a boat leaves port on a heading of 30˚ with the automatic pilot set for 10 knots. On this particular day, there is a 5-knot current with a heading of 60˚. The vector diagram at the right shows the effect of the current on the position of the boat at the end of one hour.

   a. Assuming a scale of 1 cm = 2 nm, verify the accuracy of the diagram.
b. The sum of the course and current vectors gives the position of the boat in one hour. Determine how far the boat will actually travel in one hour:
   ■ Using the scale diagram
   ■ Using the Law of Cosines

c. At what speed and on what heading will the boat actually travel during the first hour? Would it continue to travel similarly during the next hour if all conditions remained the same? Explain.

2. In Activity 1, you were able to determine the actual course of the boat using either a scale drawing and measurement or using the Law of Cosines. Now examine the situation in terms of component vectors.

   a. Compute the lengths of the horizontal and vertical components of the vector representing the planned course during the first hour. Sketch each vector beginning at point $P$, the initial point of the planned-course vector.

   b. Compute the lengths of the horizontal and vertical components of the current vector at the end of one hour. Sketch each component vector at the terminal point of the planned-course vector.

   c. Using the component vectors found in Parts a and b, find the components, magnitude, and heading of the resultant vector representing the actual route sailed. What assumption are you making about addition of vectors?

   d. Compare the results of this activity with those of Activity 1.

3. Consider force vectors $\vec{v}$, with length 4 cm and heading 55˚, and $\vec{u}$ with length 5 cm and heading 20˚.

   a. On graph paper, draw $\vec{v} + \vec{u}$.

   b. Draw the horizontal and vertical components of $\vec{v}$ and of $\vec{u}$.

   c. Draw the resultant of the vertical components and the resultant of the horizontal components.

   d. Draw the sum of the two resultant vectors found in Part c. How is this sum related to $\vec{v} + \vec{u}$? Explain.

   e. Describe how the components of two vectors can be used to find the sum of the two vectors.

4. Make a sketch of a vector diagram showing the location of an airplane at the end of one hour if its heading was 60˚ and its speed in still air was 500 mph, but the wind was blowing at 50 mph on a heading of 120˚.

   a. Augment your sketch to show the horizontal and vertical components of the planned-course velocity vector. Represent the horizontal and vertical component vectors of the wind velocity vector in a manner consistent with addition of vectors in this situation.

   b. Use the components of the vectors in Part a to determine the heading and distance the airplane traveled in one hour.
c. What was the effective speed of the airplane with respect to the ground? (This is called the ground speed.)

d. Describe another way to determine the distance traveled by the airplane in one hour.

The process illustrated in Activities 3 and 4, called component analysis of vectors, is a very powerful tool for analyzing linear motion problems. It reduces a complex situation to one in which only component vectors with the same direction are added.

5. Two boys have to move a doghouse on skids to a new position due east of its present location. They tie ropes to the doghouse and pull as follows: Thad pulls with a force of 100 pounds on a heading of 45°, while Jerame pulls with a force of 120 pounds on a heading of 120°.

a. Make a sketch showing the vectors involved.

b. Find the heading on which the doghouse should move under these conditions.

c. If the doghouse weighs 150 lb, will it move? Explain your reasoning.

d. How should Jerame change the heading at which he pulls so that the doghouse slides due east?

6. Mary and Kim are blockers on their respective school volleyball teams. Suppose that in a conference match, at the same time, they each hit the ball when it is directly over the net. Mary’s hit has a force of 50 pounds on a 125° heading. Kim’s hit has a force of 40 pounds on a 30° heading.

a. Sketch the vectors involved if the net is on the east-west line.

b. Assuming that the ball moves in the direction of the resultant force, on whose side of the net will the ball land? How can component vectors be used to prove this?

c. At what angle should Mary hit the ball so that it follows the top of the net or goes onto Kim’s side?

7. In Activity 1, you found that a current in the water causes a boat to travel on the resultant of two forces. Recall that the boat was set to travel on a 30° heading at 10 knots and the current was flowing on a 60° heading at 5 knots. Use this information to find the vector that needs to be added to the current vector \( \vec{c} \) to give the course vector \( \vec{b} \). This new vector \( \vec{x} \) is the path the boat needs to steer so it follows the desired course.
In this investigation, you examined how vectors can be used to model situations in which more than one force is acting on an object.

a. Describe how vector models can be used to model linear motion in moving air or water.

b. Explain how the horizontal and vertical components of vectors can be used to determine heading and speed of a boat or airplane that is moving in water or air that is also moving.

Be prepared to share your descriptions and thinking with the entire class.

On Your Own

A commercial jet airplane cruises at 600 mph in still air. The pilot wants to fly on a heading of 20° and average 600 mph, but a 70 mph wind is blowing from the northwest (a heading of 135°).

a. Draw a vector model of the effect of the wind on the jet.

b. Draw a vector model showing the heading needed to keep the jet on course and compute the heading.

c. Compute the still air speed that the jet needs to maintain to attain the desired average of 600 mph.

INVESTIGATION 4 Coordinates and Vectors

The vectors you have used up to now have been located in a north/east coordinate system. Because the direction north can be found by sighting a star, using a compass, or by using a Global Positioning System (GPS), navigation both on water and in the air describes the direction a vector points (its heading) in terms of degrees clockwise from due north. Mathematicians and many scientists use a different way to describe the direction of a vector. When using a rectangular coordinate system, the direction of a vector is measured by the angle the vector makes with the positive x-axis with the angle measured counterclockwise. In this
investigation, you will explore some of the advantages of representing vectors in a standard \((x, y)\) coordinate system.

1. Sketch vectors satisfying the criteria given. Re-express headings as directions and directions as headings.
   
   a. \( \overrightarrow{v} \) has length 4 cm and heading 80°.
   
   b. \( \overrightarrow{p} \) has length 5 cm and direction 80°.
   
   c. \( \overrightarrow{m} \) has length 3 cm and direction 130°.
   
   d. \( \overrightarrow{n} \) has length 2 cm and heading 130°.

2. Describe and illustrate the differences between a vector with heading 200° and a vector with direction 200°. For what angles are the heading and direction of a vector identical?

3. A vector with magnitude 6 and direction 80° is represented on the coordinate system below.

   a. On a copy of this coordinate system, carefully draw the following additional vectors.
      
      - magnitude 4, direction 145°
      - magnitude 3, direction 240°
      - magnitude 5, direction 315°

   b. Sketch the horizontal and vertical components of each of the four “parent” vectors given above. Estimate the coordinates of the terminal point of each component vector. How are these coordinates related to the coordinates of the terminal point of the given “parent” vector?

   c. Consider the vector with magnitude 6 and direction 80°. Write equations that express the coordinates \((x, y)\) of the terminal point of the vector in terms of its magnitude and its direction. Explain how the magnitudes of the components can be found using these equations.

   d. Using a method similar to that in Part c, compute the coordinates of the terminal points of the three vectors in Part a. Compare the coordinates with the estimates you made in Part b.
4. Suppose a coordinate system is placed on a nautical map so that the Grand Haven Marina is located at the origin. A speedboat leaves the marina at a direction of 30˚ and proceeds at 18 knots.

a. Make a sketch on the coordinate system showing the path of the boat.

b. Determine the coordinates of the boat’s position on the path at $\frac{1}{2}$ hour, 1 hour, and 2 hours.

c. Write rules giving the coordinates $(x, y)$ of the position of the boat for any time $t$ (in hours).

5. Suppose the speedboat in Activity 4 traveled at a direction of 120˚ rather than 30˚.

a. Sketch the path of the boat on the same coordinate system and identify its position at $\frac{1}{2}$ hour, 1 hour, and 2 hours.

b. What are the vector components of each of the positions in Part a?

c. Write rules giving the coordinates of the position of the boat for any time $t$ (in hours).

d. Repeat Part c if the direction is 210˚ and if the direction is 330˚.

e. Compare the rules you wrote for Parts c and d of this activity and for Part c of Activity 4.

6. Now consider how a standard coordinate system can be used to model straight line motion of an aircraft. Suppose a commercial jet leaves New York City and flies at a direction of 190˚ towards the West Coast at 600 mph.

a. Model this situation by placing New York City at the origin of a coordinate system and sketch the aircraft’s path westward.

b. Find rules for the coordinates of the position of the aircraft $t$ hours into the flight. Then find the coordinates of the aircraft’s position after 0.25 hour, 1.3 hours, and 3.2 hours.

c. Find the coordinates of the aircraft’s position when it has flown 2,000 miles.
7. Two families of hikers leave a base camp on a mesa with directions of 31° and 42° respectively. Because of the ages of family members, the first family averages about 0.8 mph while the second family averages 1.1 mph.

a. Sketch the hiking paths of the two families on a standard coordinate system. Assume the families continue to hike in the directions they started and at the indicated rates.

b. At the end of one hour, what are the coordinates of their positions? How far apart are they?

c. How far apart are the families after 2 hours? After 3 hours?

d. How far apart are the families after 2 hours? After 3 hours?

e. After how much time will they be about 3 miles apart?

8. Two tugboats are maneuvering a supply barge into a slip. (A slip is a docking place for a boat.) One tugboat exerts a force of 1,500 pounds with direction of –20°; another exerts a force of 2,000 pounds with direction 70°.

a. Use a coordinate system to sketch the situation.

b. Find the direction and magnitude of the resultant force on the barge.

Checkpoint

In this investigation, you discovered some of the advantages of representing vectors in a rectangular coordinate system.

a. Suppose \( \overrightarrow{OB} \) is a vector whose initial point is at the origin. If \( B \) has coordinates \((5 \cos 135°, 5 \sin 135°)\), what is the length of \( \overrightarrow{OB} \)? What lines or segments determine the sides of the angle that has measure 135°?

b. Describe the relationships among a vector, its component vectors, and the coordinates of the terminal point when the initial point of the vector is at the origin.

Be prepared to explain your responses to the entire class.
A vector with its initial point at the origin of a coordinate system is said to be in **standard position** and is called a **position vector**. The coordinates of the tip of a position vector are the coordinates of the point where the vector ends. Every vector in a given coordinate system is equal to some position vector in that system. Since the tip of every position vector has unique coordinates \((x, y)\), the ordered pair is often called the vector. That is, if \(\vec{a}\) is a position vector, then \(\vec{a} = (x, y)\). Recall that \(\vec{a} = [r, \theta]\), where \(r\) is the magnitude and \(\theta\) is the direction. Thus, there are at least three ways to represent vectors.

**On Your Own**

Suppose a cruise liner is experiencing mechanical problems 4 hours out of a harbor at a direction of 110˚. During these 4 hours, the ship averaged 35 knots on its 110˚ course.

**a.** Using a standard coordinate system with the origin as the launch point, how far is the ship from its launch after 4 hours?

**b.** Find the coordinates of the position of the cruise liner.

**c.** In addition to the harbor from which the ship departed, there are ports located at \((-75\, \text{nm}, 75\, \text{nm})\) and \((25\, \text{nm}, 125\, \text{nm})\). Which port is closest to the ship’s position?

**d.** What conditions in addition to distance might the pilot consider in deciding which port to go to for repairs?
1. Tony Hillerman is a mystery writer whose books are often based on the native American cultures of New Mexico, Utah, Colorado, and Arizona. The map below shows Hillerman country in which Navajo Tribal Police Officers Joe Leaphorn and Jim Chee solve mysteries. In Hillerman’s novels, they travel by car throughout the reservations, but for this task assume they have a helicopter.

   ![Map of Hillerman country showing Navajo Tribal Police Officers Joe Leaphorn and Jim Chee solving mysteries.](image)

   **a.** Suppose Jim and Joe are stationed at Shiprock. What heading should Jim chart to go to Tuba City to investigate a hit-and-run accident? What is the distance he must fly by helicopter?

   **b.** Jim is to fly from Tuba City to Flagstaff to meet with FBI officials. What is his heading? At 100 mph, what is his flying time?

   **c.** Plot the round trip from Shiprock to Round Rock to Window Rock to Standing Rock and back to Shiprock. Give the heading and distance of each part of the trip.
2. Refer to the nautical chart on the right of a small portion of the Massachusetts coast. The Open C is located just off the flashing red light at Sunken Ledge when its skipper learns that fishing action has begun near the “GP” buoy.

a. What heading should the skipper set for the “GP” buoy?

b. At 6 knots, how long would the trip take in still water?

c. Now suppose there is a heavy wind with heading 190˚ that will move boats at a rate of about 2 knots. Make a vector diagram showing the effect of the wind on the course of the Open C.

d. In the wind, what is the heading of the route the Open C actually travels?

e. What heading should the skipper plot to account for the wind and arrive at the “GP” buoy?

3. Jim Chee, a helicopter pilot, wants to fly from Shiprock to Dinnebito in the Hopi-Navajo joint-use area. Using the map provided in Task 1, answer the following questions.

a. What heading should he plan?

b. If he leaves at 10:00 A.M. and travels at 100 mph, when will he arrive at Dinnebito?

c. There is a 20-mph wind with heading 150˚. Where will Jim be at his estimated time of arrival if he makes no correction for wind? How far is this from Dinnebito?

d. What course should Jim plan that accounts for the wind and ensures ending up at Dinnebito at the arrival time calculated in Part b?
4. Two boaters leave Ludington, Michigan, at 8:00 A.M. One is heading for Manitowoc, Wisconsin, on a heading of 280°. The other heads for Milwaukee on a heading of 230°. Manitowoc is about 61 miles from Ludington; Milwaukee is about 97 mi. The radios on the boats are good for distances up to 50 mi.

a. With Ludington as the origin, set up a coordinate system. What are the coordinates of Manitowoc and Milwaukee?

b. Sketch a vector diagram if the boat to Manitowoc travels at 8 mph and the boat to Milwaukee travels at 10 mph. How far from each other are they at 9:00 A.M.? At 11:00 A.M.?

c. At about what time will they lose radio contact?

d. How far from their destinations are the boats when they lose radio contact?

**Organizing**

1. Recall that for any two numbers \( s \) and \( t \), \( s - t = s + (-t) \). For example, \( 7.2 - 3.4 = 7.2 + (-3.4) \).

   a. State in words how you would subtract one number from another using this definition.

   b. By analogy to subtraction of numbers, write a verbal description of how to subtract \( \vec{b} \) from \( \vec{a} \). Use your description to sketch \( \vec{a}, \vec{b}, \) and \( \vec{a} - \vec{b} \).

   c. Complete the statement: \( \vec{a} - \vec{b} = \) __________.

   d. What would you expect to get if you added \( \vec{b} \) to \( \vec{a} - \vec{b} \)? Draw a sketch to support your conjecture.
2. On a piece of paper, mark a point \( O \) in the center. Using \( O \) as the beginning point, accurately draw a 1-inch vector, \( \overrightarrow{a} \), pointing to the right and accurately draw a 1.5-inch vector, \( \overrightarrow{b} \), pointing straight upward.

a. What is the measure of the angle between these vectors?

b. Choose a point \( P \) so that the length of \( \overrightarrow{OP} \) is 4 inches. Find scalars \( m \) and \( n \) so that \( m\overrightarrow{a} + n\overrightarrow{b} = \overrightarrow{OP} \).

c. If \( Q \) is any other point, can you always find scalars \( m \) and \( n \) such that \( m\overrightarrow{a} + n\overrightarrow{b} = \overrightarrow{OQ} \)? Explain your reasoning.

d. Discuss your answers in Part c for cases where \( \overrightarrow{OQ} \) has the same direction as \( \overrightarrow{a} \) or \( \overrightarrow{b} \), or where point \( Q \) coincides with point \( O \)?

3. Scalar multiplication is used to multiply a vector by a number. Which of the following statements about scalars, vectors, and their products are true? Explain your reasoning.

a. \( m(n\overrightarrow{a}) = (mn)\overrightarrow{a} \)

b. \( (m + n)\overrightarrow{a} = m\overrightarrow{a} + n\overrightarrow{a} \)

c. \( m(\overrightarrow{a} + \overrightarrow{b}) = m\overrightarrow{a} + m\overrightarrow{b} \)

d. The length of \( (m + n)\overrightarrow{a} \) equals the length of \( m\overrightarrow{a} \) plus the length of \( n\overrightarrow{a} \).

4. Suppose \( \overrightarrow{OA} \) and \( \overrightarrow{OB} \) are position vectors. The length of \( \overrightarrow{OA} \) is \( r \) and its direction is \( \theta \); that is, \( \overrightarrow{OA} = [r, \theta] \). \( \overrightarrow{OB} \) has length \( b \) and direction \( \phi \) (Greek letter “phi”) or \( \overrightarrow{OB} = [b, \phi] \).

a. What are the components of \( \overrightarrow{OA} \) and \( \overrightarrow{OB} \)?

b. What are the coordinates of \( A \) and \( B \)?

c. Find the components of \( \overrightarrow{OA} + \overrightarrow{OB} \).

d. What are the components of \( -\overrightarrow{OA} \)?

e. How could you define \( \overrightarrow{OB} - \overrightarrow{OA} \)?

5. Suppose you are given position vectors \( \overrightarrow{a} = (2, 3) \) and \( \overrightarrow{b} = (-1, 4) \).

a. Find the lengths and directions of \( \overrightarrow{a} \) and \( \overrightarrow{b} \).

b. Explain how you can find \( \overrightarrow{a} + \overrightarrow{b} \) using the coordinate representations of \( \overrightarrow{a} \) and \( \overrightarrow{b} \).

c. Suppose \( \overrightarrow{a} = (x_1, y_1) \) and \( \overrightarrow{b} = (x_2, y_2) \).

- What would be the coordinate representation \( \overrightarrow{a} + \overrightarrow{b} \)? Explain your reasoning.
- Explain why the definition of scalar multiplication \( m\overrightarrow{a} = (mx_1, my_1) \) makes sense.

d. Using general coordinate representations \( \overrightarrow{a} = (x_1, y_1) \) and \( \overrightarrow{b} = (x_2, y_2) \):

- Prove \( \overrightarrow{a} + \overrightarrow{b} = \overrightarrow{a} + \overrightarrow{b} \).
- Prove \( m(\overrightarrow{a} + \overrightarrow{b}) = m\overrightarrow{a} + m\overrightarrow{b} \).
1. For two vectors to be equal, two conditions must be met: their lengths must be equal and their directions must be equal.
   a. Sketch vectors to illustrate the necessity of both conditions:
      - Show that two vectors with the same length may not be equal.
      - Show that two vectors with the same direction may not be equal.
   b. If two vectors are equal and begin at the same point, how are their geometric representations (arrows) related?
   c. If two vectors are equal and begin at different points, how are their geometric representations related?

2. In each of the diagrams below, a figure \( F \) and its image \( G \) under a translation are shown.
   
   a. How could you use vectors to describe these translations?
   b. Can every translation be described by a vector? Explain your reasoning.

3. Suppose the coordinates of the terminal point of a position vector are \((a, b)\).
   a. How could these numbers be used to calculate the length of the vector?
   b. How do the signs (+ or –) on each coordinate help you sketch the vector? For example, if \(a\) and \(b\) are both negative, in which quadrant would you draw the vector?
   c. How can the coordinates \((a, b)\) be used to determine the direction of the vector? When you calculate the direction of a vector, does the calculator always give you correct directions for all combinations of signs (+, –) for \(a\) and \(b\)? Experiment to discover patterns. How can you determine the correct direction?

4. In Organizing Task 1, you examined how to subtract one vector from another. Using the diagram below, write each expression in a simpler form.
   a. \( \vec{b} - \vec{a} \)
   b. \( \vec{w} - \vec{a} \)
   c. \( \vec{v} + \vec{a} - \vec{b} \)
   d. \( \vec{b} + \vec{a} - \vec{v} - \vec{w} \)
5. Make a table of the heading and the direction of a vector as it rotates through headings from 0˚ to 360˚ in steps of 20˚. Make a scatterplot of the (heading, direction) data. Describe patterns you see in the scatterplot. Can these patterns be described algebraically?

### Extending

1. In Organizing Task 5, you may have discovered the conventional method for addition of vectors expressed in coordinate form: 
\[(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\]
Investigate and compare properties of addition of vectors expressed in coordinate form with properties of addition of 2 \(\times\) 2 matrices. Write a summary of your findings.

2. In Course 3, Unit 4, “Shapes and Geometric Reasoning,” you may have used properties of similar triangles to prove the Midpoint Connector Theorem: If a line segment joins the midpoints of two sides of a triangle, then it is parallel to the third side and its length is one-half the length of the third side.

   a. Doris claims that this theorem could also be proved using vectors as shown. Explain why her proof is or is not valid.

   Doris’s proof:

   If \(X\) and \(Y\) are the midpoints of \(\overline{AC}\) and \(\overline{BC}\) respectively,

   then \(\overline{AX} = \overline{XC} = \frac{1}{2} \overline{AC}\)
   and \(\overline{CY} = \overline{YB} = \frac{1}{2} \overline{CB}\).

   Also, \(\overline{AC} + \overline{CB} = \overline{AB}\)
   and \(\overline{XC} + \overline{CY} = \overline{XY}\).

   So, \(\frac{1}{2} \overline{AC} + \frac{1}{2} \overline{CB} = \overline{XC} + \overline{CY}\) or \(\frac{1}{2} (\overline{AC} + \overline{CB}) = \overline{XY}\).

   Therefore, \(\frac{1}{2} \overline{AB} = \overline{XY}\).

   It follows that \(\overline{XY} = \frac{1}{2} \overline{AB}\) and \(\overline{XY} \parallel \overline{AB}\).

   b. Would the above argument need to be modified if \(\triangle ABC\) was an obtuse triangle? Explain your reasoning.

3. Refer to the nautical chart of the Stone Harbor, Massachusetts region on the next page. Suppose the Angler and the Free Spirit leave the mouth of the channel at Stone Harbor together. Their headings are 35˚ and 20˚ respectively. The Angler travels at 4 knots and after 30 minutes sights the Free Spirit to the north and west. The line of sight makes an angle of 110˚ with the path of the Angler from the harbor.
a. Draw the situation to scale.
b. Estimate the distance between the boats using the scale drawing.
c. Can vector component analysis be used to determine the distance? Explain your reasoning.
d. Use the Law of Sines to determine the distance between the two boats. Compare this distance to your estimate in Part b. At what speed is the Free Spirit traveling?

4. In landscaping an industrial park, a large boulder was to be moved by attaching chains to two tractors that would pull at an angle of 75° between the chains. If one tractor can pull with 1.5 times the force of the other, and the boulder requires a force of 10,000 newtons to be moved, what force is required from each tractor?

5. Every plane vector is equal to a position vector, so every two distinct vectors \( \vec{a} \) and \( \vec{b} \) determine an angle as shown below. Let \( \vec{a} = (x_1, y_1) \) and \( \vec{b} = (x_2, y_2) \).

![Diagram of vectors](image)

a. Use the Law of Cosines to write a formula for finding the angle between \( \vec{a} \) and \( \vec{b} \).
b. The numerator, \( x_1x_2 + y_1y_2 \), of the expression for the cosine of the angle determined by \( \vec{a} \) and \( \vec{b} \) is called the **inner product** or **dot product** of the vectors \( (x_1, y_1) \) and \( (x_2, y_2) \). When the angle between two vectors is 90°, what is the value of the inner product? Explain.
c. Find the angle between each pair of vectors.
- \((2, 3)\) and \((4, -3)\)
- \((-2, 1)\) and \((3, -5)\)
- \((-1, -5)\) and \((-3, -2)\)
- \((1, 2)\) and \((3, 4)\)
1. If 2 out of 3 registered voters will cast a ballot in an election and there are 450,000 registered voters, how many people will cast a ballot?
   (a) 225,000  (b) 660,000  (c) 675,000  (d) 900,000  (e) 300,000

2. Simplify \( |3 - 5| - |7 - 10| \).
   (a) -1  (b) 5  (c) 1  (d) -6  (e) -5

3. Solve simultaneously for \( x \) and \( y \):
   \( -2x + 4y = -6, \ x + 3y = -12 \).
   (a) (-1, 7)  (b) (3, 3)  (c) (-3, 3)  (d) \( \left(18, \frac{18}{7}\right) \)  (e) (-3, -3)

4. Which of the following could be a portion of the graph of a function \( y = f(x) \)?
   (a) \[ \text{Graph Image} \]  (b) \[ \text{Graph Image} \]  (c) \[ \text{Graph Image} \]
   (d) \[ \text{Graph Image} \]  (e) \[ \text{Graph Image} \]
5. If \(x^2 - x = 12\), then the smaller solution is
(a) \(-4\)    (b) \(1\)    (c) \(-3\)    (d) \(-2\)    (e) \(3\)

6. If \(11 - 5x > 7\), then
(a) \(x < 3.6\)    (b) \(x < -0.8\)    (c) \(x > 3.6\)    (d) \(x < 0.8\)    (e) \(x > 0.8\)

7. The graph of a function \(y = f(x)\) is shown. Which of the following is true?
(a) \(f(0) > 0\)    (b) \(f(1) > f(-4)\)    (c) \(f(-1) < f(-2)\)    (d) \(f(-3) > f(0.5)\)    (e) \(f(-5) < f(0)\)

8. In the figure shown below, the perimeter of the square is 8. What is the area of the inscribed circle?
(a) \(16\pi\)    (b) \(\sqrt{8}\pi\)    (c) \(4\pi\)    (d) \(\pi\)    (e) \(4 - \pi\)

9. Simplify \(3x^3(x^2 + x^3)\).
(a) \(-3x\)    (b) \(-\frac{3}{x} + 3\)    (c) \(\frac{3}{x} + 3\)    (d) \(\frac{3}{x} + 3x\)    (e) \(\frac{3}{x^6} + \frac{3}{y^2}\)

10. Rewrite using radical notation: \(3x^{\frac{1}{2}}y^{\frac{3}{2}}\)
(a) \(3\sqrt[2]{x}\sqrt[2]{y}\)    (b) \(\sqrt[2]{3x^2y^3}\)    (c) \(3y\sqrt[2]{x}\)    (d) \(3\sqrt[2]{xy^2}\)    (e) \(\sqrt[2]{3xy^2}\)
In Lesson 1, you modeled linear motions of boats and airplanes. There are many other kinds of motions that are nonlinear. The wheels of an automobile or bicycle rotate around their centers. Satellites orbit Earth. A catcher for a softball or baseball team with a weak arm throws a “rainbow” to second base. A tennis player serves a ball so it just clears the net and lands in the service court. A golfer drives a ball over 300 yards to the middle of the fairway. You and a group of friends may play volleyball or compete in a friendly game of darts. In each of these contexts, both direction and distance are important elements.

Think About This Situation

Carnivals and county fairs often include games in which you throw a baseball at a pyramid of bottles or at a target. Imagine that a target is 10 meters away. Your goal is to hit it with a ball.

a. What are important variables that may affect the outcome of your throw?

b. If you can throw hard, where should you aim?

c. Where should you aim next if your first throw just barely makes it to the target?

d. What are some factors that affect the path of the ball?

In this lesson, you will learn methods for simulating linear and nonlinear motions. By building graphing calculator or computer-based simulation models, you will be able to see the paths traveled by moving objects. You will even be able to simulate races and Ferris wheel rides. You will begin by simulating motion along a line.
Parametric Models for Linear Motion

In Lesson 1, you learned how the coordinates of the terminal point of a vector can be determined by finding the components of the vector. In modeling the motion of a boat, the terminal point of a vector identifies the location of the boat. Motion involves change in location over time. A boat, for example, is at different places 15 minutes and 20 minutes into a trip. In Activities 1–3, you will investigate how to write rules giving the location of a moving object in terms of the time it has been moving, its elapsed time.

1. Suppose the Wayfarer begins at the origin of a coordinate system and follows a course with direction (not heading) $60^\circ$ and a speed of 8 knots.

   a. Sketch the path of the Wayfarer on a coordinate system.

   b. Write rules for the horizontal and the vertical components of a point on the ship’s path in terms of elapsed time in hours ($t$).

   c. A partial table of values for elapsed time and the corresponding horizontal and vertical components is shown at the right. Use your rules to complete a copy of the table for values of $t$ up to 2.0 hours.

   d. Describe how $t$ changes; how $x$ changes; and how $y$ changes. What are the units of measure for $x$ and $y$?

   e. Make a scatterplot of the $(horizontal \ component, vertical \ component)$ data. Describe the pattern in the plot. Explain why this pattern makes sense.

2. Most graphing calculators and computer graphing software have a parametric function capability that enables you to quickly construct a table like the one above. Set your calculator or software to accept angle measures in degrees and parametric equations and, to display multiple graphs simultaneously in dot (not connected) format. The MODE screen to the left of one popular graphing calculator shows the correct settings. You may need to set graph styles differently on your calculator or software.
Now choose the \( \text{Y=} \) menu. Notice that the equations are paired. For the first pair, enter your rules for the \( x \) and \( y \) components from Activity 1 Part b. Your display should be similar to one of the two screens below.

![Graphs](Plot1, Plot2, Plot3)

\[
\begin{align*}
X1T &= 4T \\
Y1T &= 6.92820323T \\
X2T &= Y2T \\
X3T &= Y3T \\
X4T &= Y4T
\end{align*}
\]

a. Why do both of these displays represent how \( x \) and \( y \) change with respect to time \( t \)?

b. Now use the table-building capability of your calculator or computer software to generate a table for \( T \), \( X_{IT} \), and \( Y_{IT} \) beginning at 0 with 0.1 increments in \( T \). Compare this new table with the one you completed in Part c of Activity 1. Do the patterns you noted in Part d of that activity continue in this more extensive table?

Once you have the rules for \( X_{IT} \) and \( Y_{IT} \) entered in your calculator or software, you can display a graph of the model for the path of the ship. As with other graphical displays, you need to first set the viewing window. The settings shown below do the following:

- Since the independent variable here is \( T \), \( \text{Tmin} = 0 \) sets the calculator to begin evaluating \( X_{IT} \) and \( Y_{IT} \) at \( T = 0 \).
- \( \text{Tstep} = 0.1 \) increments \( T \) by 0.1 at each step until \( T \) is larger than \( \text{Tmax} \).
- The X and Y settings establish the lower and upper bounds of the viewing screen.

3. Set your calculator or software to conform to the above conditions and then plot the \((x, y)\) pairs from Activity 2.

a. Compare your display with the one shown at the right. If they differ, check each menu and your graph style settings.

b. Experiment with various ranges and step sizes for \( t \). Try to answer questions such as the following. How many points are displayed? Why? What settings will show 21 points? Does the graph need to begin at the origin? How can you make it begin at a different point? How can you make sure all the points are displayed on the screen?

c. Explore your display by tracing along it. Compare the values of \( t \), \( x \), and \( y \) shown on the screen with those in your table.
The rules you used to generate the coordinates for the terminal point of the vector locating the Wayfarer at any time \( t \) are called **parametric equations** and the variable \( t \) is called a **parameter**. In this case, your parametric equations were

\[
\begin{align*}
x &= 8t \cos 60^\circ \\
y &= 8t \sin 60^\circ
\end{align*}
\]

or

\[
\begin{align*}
x &= 4t \\
y &= 6.92820323t
\end{align*}
\]

In the next several activities, you will explore ways to use parametric equations to model linear motion in different situations.

4. Suppose a commercial jet leaves Los Angeles International Airport on a course with direction 15°. The aircraft is set to travel at a speed of 600 mph in still air.
   
   a. Develop a parametric equation model for the location of the plane after \( t \) hours if the airport is placed at the origin of the coordinate system.
   
   b. How far east has the plane traveled in 2.5 hours? How far north?
   
   c. Simulate the motion of the aircraft on the calculator screen by showing its position every half hour until it reaches the East Coast, about 3,100 miles east of Los Angeles. Describe your viewing window settings.
   
   d. How would you modify your simulation to show the location of the plane every 12 minutes?

5. Now consider how parametric equations could be used to simultaneously model the motion of two boats. Begin by turning off the axes on your graphics screen.
   
   a. Suppose the Charlotte Rose leaves its anchorage at noon going due east at 8 knots. Write parametric equations for this motion.
   
   b. Find the time needed to travel 50 nm. Display the 50 nm path on your graphics screen. Trace your graph.
   
   c. The Lady Anna begins at 1:00 P.M. from a harbor 0.4 nm north of Charlotte Rose’s starting location, traveling due east at 10 knots. Write parametric equations for this motion. Display the path.
   
   d. Devise a way to display the paths of both the Charlotte Rose and the Lady Anna so that you can see the boats moving simultaneously, with the Charlotte Rose departing first.
   
   e. Use your display to decide which boat travels 50 nm first. Where is the other boat when the first boat has gone 50 nm? At what time will they have traveled the same distance?
f. How would you change your equations if each boat were traveling due west of the original starting point for 50 nm? Make and test the changes.

6. In previous courses, you developed the ability to predict the shape of the graph of various functions by examining their symbolic rules. This activity will help you extend your symbol sense to parametric equations. For each pair of parametric equations:

- Predict what the graph will look like.
- Check your prediction using your calculator or graphing software.
- Make a sketch of the displayed graph and label it with the corresponding pair of equations.

For each case, use $T_{min} = 0$, $T_{max} = 4$, and $T_{step} = 0.1$. The viewing window should be $-40 \leq X \leq 40$ and $-40 \leq Y \leq 40$, with $X_{scl} = 10$, $Y_{scl} = 10$.

- **a.** $x = 5t$  
  $y = 8$
- **b.** $x = -5t$  
  $y = 8$
- **c.** $x = 6(t - 2)$  
  $y = -20$
- **d.** $x = -7(t + 1)$  
  $y = 10$
- **e.** $x = 10$  
  $y = 5t$
- **f.** $x = 10$  
  $y = -5t$
- **g.** $x = -20$  
  $y = 6(t + 2)$
- **h.** $x = 8$  
  $y = -4(t - 5)$

7. Look back at the parametric equations and graphs you produced in Activity 6.

- a. What general patterns do you see relating the shape and placement of a graph to the symbolic form of the equations?

- b. Write a pair of parametric equations different from those in Activity 6, but with the same shape graph. Trade equations with a partner. Predict what the graph of your equations will look like and then test your prediction. If either your prediction or that of your partner is incorrect, identify the possible cause of the error and then repeat for a different pair of equations.

- c. Explain why the following parametric equations have the same graph as the parametric equations in Part a of Activity 6.
  
  $x = 5t \cos 0^\circ$
  
  $y = 5t \sin 0^\circ + 8$

- d. Explain why the following parametric equations have the same graph as the parametric equations in Part e of Activity 6.
  
  $x = 5t \cos 90^\circ + 10$
  
  $y = 5t \sin 90^\circ$

- e. Determine which of the remaining pairs of parametric equations in Activity 6 can be represented in the form below. Explain your reasoning.
  
  $x = At \cos \theta + B$
  
  $y = At \sin \theta + D$
8. Test your understanding of modeling with parametric equations in the following situation: Suppose the cabin cruiser *Sawatdee* begins at noon heading due east at 8 knots. The *Delhi Dhaba* begins at noon at a location 60 nm due east of the *Sawatdee* and heads due west at 10 knots.

   a. Represent this situation on a coordinate system.
   b. Write parametric equations for each motion. *(Hint: When $t = 0$, what is the $x$-value for the *Sawatdee*? For the *Delhi Dhaba*?)*
   c. At what location and time will they meet?

You may recall from your work with quadratic models in previous courses that when an object is dropped from a height above the surface of Earth, the velocity and distance traveled are functions of gravity and time. Neglecting air resistance, the average velocity after $t$ seconds is $-4.9t$ meters per second or $-16t$ feet per second. Thus, the directed distance an object falls in $t$ seconds due to gravity is $(-4.9t \text{ meters/second})(t \text{ seconds}) = -4.9t^2$ meters. In feet, the directed distance traveled is $-16t^2$ feet.

9. Suppose an object is released from a weather balloon 200 meters above the surface of Earth.

   a. Explain why the height, in meters, of the object above the surface of Earth after $T$ seconds can be represented by these parametric equations:

   $\begin{align*}
   X_T &= 1 \\
   Y_T &= -4.9T^2 + 200
   \end{align*}$

   b. Select appropriate values for $T$ and a viewing window to display the motion of the object. Use dot mode. Sketch the display.
   c. For a specific value of $T$ (such as 1.34 seconds), what does the corresponding value of $Y_T$ tell you about the object?
   d. Describe how the distance the object falls per second changes with increasing time. How can this be observed in the graph? In a table?
   e. How many seconds after the drop does the object strike the Earth? How did you determine the time?
   f. In this example $X_T = 1$. Is it important that $X_T$ be 1, or could it be another number? Explain your reasoning.
   g. Write a pair of parametric equations that describes the height, in feet, of an object that is dropped from a point 150 feet above the Earth’s surface.
In this investigation, you explored how parametric equations can be used to model linear motion.

**a** How do parametric equations differ from other algebraic equations you have studied?

**b** How are parametric equations of a point moving along a line related to vectors?

**c** Describe how you would write parametric equations of:
  - A horizontal linear path at a constant velocity
  - A vertical linear path at a constant velocity
  - An oblique linear path through the origin at a constant velocity

*Be prepared to share your ideas and descriptions with the class.*

The general parametric equations for linear motion with a constant velocity are

\[
\begin{align*}
x &= At \cos \theta + B \\
y &= At \sin \theta + D
\end{align*}
\]

When \( \theta = 0^\circ \), these simplify to \( x = At + B \) and \( y = D \). When \( \theta = 90^\circ \), these simplify to \( x = B \) and \( y = At + D \).

**On Your Own**

Average velocity, after \( t \) seconds, of a falling object differs among celestial bodies.

<table>
<thead>
<tr>
<th></th>
<th>Earth</th>
<th>The Moon</th>
<th>Jupiter</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-4.9t \text{ m/s})</td>
<td>(-0.83t \text{ m/s})</td>
<td>(-11.44t \text{ m/s})</td>
<td></td>
</tr>
</tbody>
</table>
a. Suppose an object is dropped from 100-meter tall structures on Earth, on the Moon, and on Jupiter. Write parametric representations of each motion.

b. Display the motion of the objects in the same viewing window. Describe differences and similarities in the patterns of change.

c. Find the time it takes for the object to strike the surface of each body.

d. Draw the vector that shows the position of the object falling on the Moon at \( t = 4.5 \). What is the \( y \)-component of this vector? What does it tell you?

### INVESTIGATION 2: Parametric Models for Nonlinear Motion

You have seen that linear motion can be described in terms of two components, a horizontal component and a vertical component. Most objects such as boats, trains, or airplanes continue to move along a straight line because energy is applied to maintain the speed. Without this energy, other forces such as friction or gravity would cause the object to slow down and finally stop.

In the game of slow-pitch softball, the pitcher throws the ball underhanded so that it goes high in the air and crosses the plate as it comes down. The batter tries to hit it as it comes down.

1. Consider a pitcher on a slow-pitch softball team who throws the ball toward home plate at a speed of 12 meters per second at an angle of 55˚ with the horizontal. The pitcher tries to get the ball to drop nearly vertically across the plate.

   a. For the moment, assume that the speed of the ball is constant at 12 m/s. Represent this situation on a coordinate system. Then sketch the vector showing the position of the ball after 1 second. Sketch the components.

   b. Write parametric equations describing the position with respect to time of a ball pitched at an angle of 55˚ to the ground with a constant velocity of 12 m/s and without the effect of gravity. What are the units of the parameter \( t \) in the equation?

   c. Display the graph using your graphing calculator or computer software.

   d. What are some limitations of this model?

2. Recall that the effect of gravity on the position of a falling object is represented by a vector \( 4.9t^2 \) meters long and pointing straight down (\( 4.9t \times t \sin 270^\circ \) or \( 4.9t \times t \sin (-90^\circ) \)). In this activity, you will refine your model in Activity 1 to take into account the effect of gravity on the softball pitch.

   a. Modify your parametric equations for the softball pitch to include the gravitational component.

   b. Sketch the vector that shows the position of the ball at each elapsed time.

      - 1 second
      - 1.7 seconds
      - 3.3 seconds
c. Now use your graphing calculator or software to simulate the motion of the pitched ball and investigate more closely the motion of the ball.

- How long is the ball in the air?
- What is its maximum height and when does it occur?

d. In slow-pitch softball, the pitcher stands on a pitching rubber 13.7 m from home plate. Will the pitched ball make it to the plate?

e. What are some limitations of this refined model?

3. Conduct the following experiments to help refine your model of the motion of the pitched ball in Activity 2.

a. **Experiment 1:** Physically simulate tossing a slow-pitch softball. Estimate the height above the ground at which you would release the ball. Use your result to modify the model of the slow-pitch toss.

b. **Experiment 2:** Use chalk or a piece of tape to simulate the front edge of the pitching rubber. Standing with both feet on the pitching rubber, step forward and simulate pitching a softball. Estimate the distance in front of the pitching rubber that you release the ball. Use your result to further modify the model to account for the distance in front of the pitching rubber that the ball is released.

c. Use your modified parametric model to estimate the height of the pitch when it passes over home plate. Should it be called a strike?

d. For a player of average height, the strike zone is between 0.5 m and 1.5 m above the ground. Modify the initial velocity of the ball until the pitch crosses the plate inside this strike zone. (Maintain the release angle at 55°.)

e. Now modify the angle of release to get the pitch across the plate inside the same strike zone when thrown with an initial velocity of 13 m/s.

f. Explain why it is difficult to consistently pitch a ball with great accuracy.

Activities 4–6 provide other contexts that can be modeled using parametric equations. Your group should scan the three activities and then, in consultation with your teacher, select one to complete and report on to the entire class.

4. World-class horseshoe pitchers are very accurate. For example, Walter Ray Williams, a national champion, averages about 80% ringers. The horseshoe pitching court has metal stakes 40 feet apart. The stakes stand 18 inches out of the ground.

a. Walter Ray pitches a horseshoe at 45 feet per second, at a 14° angle to the ground. He releases the horseshoe at about 3 feet above the ground and 2 feet in front of the stake at one end. Write parametric equations modeling a typical throw.

b. How long is the thrown horseshoe in the air?
c. How close to 40 ft is the horizontal component when the horseshoe hits the ground?

d. If Walter Ray releases a horseshoe at 13° or 15° instead of 14°, what happens to the length of his pitch?

5. Ichiro Suzuki, a baseball player with the Seattle Mariners, was the American League Most Valuable Player for the 2001 season. When he hits the ball well, it leaves the bat at about a 29° angle, 1 meter above ground, with a velocity of 40 meters per second (ignoring wind).

ICHIRO SUZUKI

a. If the outfield wall is 6 m high and 125 m from home plate, how high will Ichiro’s hit be when the ball reaches the plane of the wall? Is it a home run?

b. How long is the ball in the air?

c. How far does the ball travel?

d. What is the maximum height the ball attains?

e. How far would the ball travel if it left the bat at a 31° angle?

6. Se Ri Pak, a professional golfer, can swing her driver at about 132 feet per second. A driver with 10° loft will propel a ball at about 150 ft/sec at an angle of about 30° (since the ball is met on the upswing and its compression adds to its velocity).

ICHIRI SUZUKI

a. Write modeling equations that describe the position of the ball for any time $t$.

b. How long is the ball in the air?

c. If the ball runs 30 to 50 yards after it hits the ground, what is the total drive length in yards?

d. If a golfer wanted to lengthen her drive, should she learn to hit the ball higher by 5° or swing the club faster by 5 ft/sec? Explain your answer.
An object thrown or hit in the air may begin on a linear path, but when additional energy is not available, other forces affect the path and it becomes curved. Look back at your work on Activities 1–3 and for the activity you chose from Activities 4–6.

**a** Name several factors that affect the path of a moving object. Which component (horizontal or vertical) does each factor affect?

**b** Explain how the horizontal and vertical components of a motion may be used to model a motion and display it graphically. What units are used for each variable?

**c** Explain how your graphing calculator or computer software must be set up to graphically display motion as a function of time.

*Be prepared to report on your chosen activity and share your group’s thinking with the entire class.*

In general, if there are several forces acting on a body, the resultant motion is the sum of the corresponding horizontal and vertical components. The parametric equations

\[
\begin{align*}
x &= At \cos \theta + Bt \cos \phi + C \\
y &= At \sin \theta + Bt \sin \phi + D
\end{align*}
\]

model the location \((x, y)\) of an object under forces acting at angles \(\theta\) and \(\phi\) in a plane with initial velocities \(A\) and \(B\), respectively. The values of \(C\) and \(D\) are the initial horizontal and vertical distances of the object from the \(x\)-axis and \(y\)-axis respectively. When the second force is gravity, which has direction 270°, these equations become

\[
\begin{align*}
x &= At \cos \theta + \frac{gt^2}{2} \cos 270° + C \\
y &= At \sin \theta + \frac{gt^2}{2} \sin 270° + D
\end{align*}
\]

where \(g\) is the gravitational constant, or equivalently

\[
\begin{align*}
x &= At \cos \theta + C \\
y &= At \sin \theta - 4.9t^2 + D \text{ (distance in meters)} \\
y &= At \sin \theta - 16t^2 + D \text{ (distance in feet)}
\end{align*}
\]

since \(\cos 270° = 0\) and \(\sin 270° = -1\).
On Your Own

Suppose Luisa begins her 10-meter platform dive with a velocity of about 2.3 meters per second. The angle at which Luisa leaves the platform is about 85˚.

a. Write parametric equations modeling Luisa’s position during the dive.

b. About how long is she in the air?

c. How high above the platform is she before she starts moving toward the water?

d. How far does she move horizontally before hitting the water?

e. Divers that push off nearly vertically could hit the platform on the way down. How many meters is Luisa from the platform when she passes it during the dive?

f. If the push-off angle were changed to 80˚, how close to the platform would she come?

INVESTIGATION 3 Representing Circles and Arcs Parametrically

One of the most common nonlinear motions you see about you is that of objects turning around a point, that is, circular motion. The wheels on cars and bicycles, disks for computers, CDs of your favorite music, drive-shafts for lawn mowers, VCR tapes, carnival rides such as Ferris wheels, and many other toys, tools, and machines use circular motion in some way.

The computer information industry uses circular motion in many of its data storage devices. On a computer disk, data is stored on tracks in sectors, as illustrated in the diagram below.
One important reason that circular disks are better than magnetic tape for recording information electronically is that the entire recording surface is accessible to the reading head almost instantaneously, simply by rotating the disk. The engineering challenge is to design a way for the reading head to identify its location on the spinning disk and, after that, where it should read information from a specific track and sector.

In the following activities, you will explore how to represent circles with parametric equations and then how to model circular motion.

1. Suppose the circle shown below has radius 6 cm. Point $P$ is on the circle and $\overline{OP}$ makes an angle of $\theta$ with the positive $x$-axis as point $P$ moves around the circle.

   ![Diagram of a circle with point P](image)

   a. How long is $\overline{OP}$?
   b. What is the direction of $\overline{OP}$?
   c. What are the components of $\overline{OP}$?
   d. Write parametric equations describing the coordinates of point $P$.
   e. Use your parametric equations and a graphing calculator or computer software to produce a graph of the circle. Is the graph what you expected? Explain. If necessary, change your window settings to display a graph of a circle.
   f. What does the parameter $T$ represent? If $T_{\text{min}} = 0$, what is the smallest $T_{\text{max}}$ value needed to produce a complete circle? Explain.

2. Investigate and then explain how you could set your calculator or computer software so that it displays only the part of the circle described in each case below.
   a. Quarter-circle between the positive $x$- and $y$-axes
   b. Half-circle to the left of the $y$-axis
   c. Half-circle below the $x$-axis
   d. Half-circle to the right of the $y$-axis
   e. Quarter-circle above the lines $y = x$ and $y = -x$
   f. An arc of a circle that a classmate describes to you
Recall that radians as well as degrees can be used to measure angles. While a degree is the measure of an angle determined by an arc that is $\frac{1}{360}$ of a complete circle, a radian is determined by an arc that is $\frac{1}{2\pi}$ of a complete circle. Since the circumference of a circle is $2\pi r$, the length of the arc corresponding to an angle of one radian is $\frac{1}{2\pi} \cdot 2\pi r = r$ linear units. The diagrams below illustrate these ideas.

3. In the following two diagrams, each ray corresponds to the terminal side of an angle whose initial side is the positive $x$-axis. Associated with each ray is the degree or radian measure of the angle. Using a copy of each diagram and the fact that $2\pi$ radians = 360˚, determine the missing angle measures. Write radian measures with fractions involving $\pi$ when appropriate.

- **a.**

- **b.**

3c. For each angle measure in Parts a and b, indicate the corresponding measure in revolutions. For example, $150^\circ = \frac{5}{12}$ revolution.

Remember that, when calculating or graphing trigonometric functions, it is very important to always check the mode. Before proceeding with the next activity, set your calculator or graphing software to Parametric and Radian modes.

4. Consider again a circle with center at the origin and radius 6 cm as in Activity 1.

- **a.** Explore how to produce the graph of this circle when $\theta$ is measured in radians. Write the parametric equations you would use.
b. What T settings for the viewing window enable you to produce a circle with the appropriate shape? Record your T settings.

c. Display a quarter-circle above the x-axis and to the right of the y-axis.

d. Display a half-circle below the x-axis.

e. Display a half-circle to the left of the y-axis.

f. Display a quarter-circle below the x-axis and to the left of the y-axis.

g. Display a quarter-circle above the line $y = x$ and above the line $y = -x$.

h. In what direction, clockwise or counterclockwise, were the graphs in Parts c–g drawn? Investigate how to draw them in the opposite direction.

Checkpoint

Consider these pairs of parametric equations:

\[
\begin{align*}
\text{x} &= 3t \cos 42^\circ \\
\text{y} &= 3t \sin 42^\circ
\end{align*}
\]

\[
\begin{align*}
\text{x} &= 3 \cos t \\
\text{y} &= 3 \sin t
\end{align*}
\]

a. Which pair of equations produces a line? A circle? Explain how you can tell by looking at the symbolic form of the rules.

b. What does “3” represent in each case?

c. What does “t” represent in each case?

Be prepared to share your group’s thinking with the entire class.

On Your Own

Write parametric equations for the circle with radius 8 and center at the origin.

a. With your calculator or computer graphing software set in Degree and Dot modes and axes turned off, adjust the viewing window so that it shows 42 dots on the circle in one revolution. Record your settings.

b. With your calculator or computer software in Radian and Dot modes, adjust the viewing window so that it shows 25 dots on the circle in one revolution. Record your settings.
Simulating Orbits

You now know how to write parametric equations for a circle and to use a graphing calculator or computer software to display the circle. In order to simulate rotating objects such as CDs, you need to draw on the idea of angular velocity. Recall that the angular velocity of a rotating object is described in terms of the degrees (or radians) through which the object turns in a unit of time. For example, 3,600 degrees per second or $20\pi$ radians per second each describe the angular velocity of an object making 10 complete revolutions per second. Thus, for a particular time, $t$, the measure of the angle through which an object has turned is

\[ \theta = 3600^\circ t \quad \text{or} \quad \theta = 20\pi t \text{ radians}. \]

Because the value of the parameter $t$ determines the size of the angle $\theta$, $\theta$ is a function of the time $t$.

1. Suppose a 2.25-inch radius CD is making 8 counterclockwise revolutions per second. To track the position of a point $P$ on the CD as a function of time in a coordinate model, assume the disk revolves about the origin $O$ and that at $t = 0$, point $P$ is at $(2.25, 0)$.

   a. Describe the angular velocity in degrees per second and in radians per second.

   b. Through how many degrees and how many radians has point $P$ turned when $t = \frac{1}{100}$ sec? When $t = \frac{1}{50}$ sec? When $t = \frac{1}{10}$ sec?

   c. Write parametric equations that give the location of point $P$ for any time $t$. Give both the degree and radian forms of the equations.

   d. How is $t$ used to determine the size of the angle of rotation?

   e. Set your graphing calculator or graphing software window appropriately: Set $T_{\min} = 0$, $T_{\max} = 1$, and $T_{\text{step}} = 0.013$, with $X_{\min} = -5$, $X_{\max} = 5$, $Y_{\min} = -3$, and $Y_{\max} = 3$. Set the mode to Dot and Radian. Using the radian form of the parametric equations, display the graph. How many times does the point rotate around the circle? Why does your answer make sense in terms of your window setting?

   f. Suppose you wanted to simulate the rotation of point $P$ around the circle exactly once. How would you change the $T$ values? How could you simulate point $P$ moving around the circle two times? Three times?

   g. Repeat Part e but with the mode setting Connected. Describe your observations.
2. Sometimes when you expect to see a circle produced by a pair of parametric equations, you see something else. Enter the following parametric equations in your calculator or graphing software:

\[ X_T = 4\cos(20\pi T) \quad Y_T = 4\sin(20\pi T) \]

Set your calculator or software to **Connected** mode. Set \( T_{\text{min}} = 0 \) and \( T_{\text{max}} = 1 \). Use \( \text{TRACE} \) to investigate each graph for differing values of \( T_{\text{step}} \). Summarize your findings.

a. Set \( T_{\text{step}} = \frac{1}{30} \). Describe the graph displayed.

b. Set \( T_{\text{step}} = \frac{1}{60} \). Describe the graph you see.

c. Set \( T_{\text{step}} = \frac{1}{16} \). Describe the graph displayed.

d. Set \( T_{\text{step}} = \frac{7}{120} \). Describe the graph you see.

e. Choose another value for \( T_{\text{step}} \) and predict the resulting graph. Check your prediction.

3. A pulley rotates counterclockwise at \( 5\pi \) radians per second. Represent the center of the pulley by the origin \( O \) of a coordinate system and let \( P \) be a point on the circumference of the pulley. The parametric equations for the terminal point of a rotation vector \( \overrightarrow{OP} \) are

\[
\begin{align*}
    x &= 7 \cos \left( 5\pi t + \frac{2\pi}{3} \right) \\
    y &= 7 \sin \left( 5\pi t + \frac{2\pi}{3} \right)
\end{align*}
\]

a. Describe the location of point \( P \) at \( t = 0 \) by giving the components of \( \overrightarrow{OP} \). What are the direction and length of \( \overrightarrow{OP} \)?

b. Describe the location of point \( P \) at \( t = 0.1, t = 0.2, t = 0.3, \) and \( t = 0.4 \) seconds.

c. Use your graphing calculator or graphing software to simulate the motion of point \( P \). Set \( T_{\text{step}} = 0.02 \).

d. How should you choose the window setting so that point \( P \) makes only one revolution?

e. How do these parametric equations differ from those in Activities 1 and 2? How are they similar?

4. A Ferris wheel with a 20-foot radius and center 24 ft above the ground is turning at one revolution per minute. Suppose that on a coordinate system, the ground surface is represented by the \( x \)-axis and the center of the wheel is on the \( y \)-axis.

a. Explain why a point on the rotating Ferris wheel can be modeled by the following pair of parametric equations:

\[
\begin{align*}
    x &= 20 \cos 2\pi t \\
    y &= 20 \sin 2\pi t + 24
\end{align*}
\]
b. Set your graphing calculator or graphing software so that it will display the motion of a rotating point for one revolution of the wheel. Record the viewing window settings you used. Explain why each setting was selected.

c. At what position on the Ferris wheel is the point located when $t = 0$?

d. How far above the ground is the seat that started at the 3 o’clock position when $t = 0.1$ minute? When $t = 0.5$ minute? When $t = 0.75$ minute?

e. How are the parametric equations modeling the position of a point on the rotating Ferris wheel different from those modeling a point on the rotating CD in Activity 1 and the pulley in Activity 3? How are they similar?

5. Once fully launched, a satellite or space station does not move in a circular orbit, but in an elliptical orbit. Ellipses will be studied more completely in Unit 8, “Space Geometry.” For now, investigate how you can modify parametric equations for a circular path to produce an elliptical path.

a. What parametric equations will stretch the circular path defined by

$$x = 8 \cos 2\pi t$$
$$y = 8 \sin 2\pi t$$

so that it has an elliptical shape similar to that in the diagram at the right? Compare your parametric equations with those of other groups.

b. Modify the parametric equations in Part a so that they produce an elliptical path that is stretched vertically.

c. Write parametric equations for an elliptical path that crosses the $x$-axis at $\pm 10$ and the $y$-axis at $\pm 5$.

d. How would you modify the equations in Part c so that the “orbit” starts at $(-10, 0)$?

6. In completing Activities 4 and 5, you probably drew on your understanding of transformations of function graphs by translating and stretching. As you saw in Course 3, those kinds of transformations are closely connected to the symbolic form of function rules. You can extend the idea of customizing function graphs to graphs of parametric equations. Consider the path modeled by the following parametric equations:

$$x = 4 \cos t$$
$$y = 4 \sin t,$$

where $0 \leq t < 2\pi$.

a. Describe the path.

b. Modify the equations so that the path is centered at $(2, 1)$.

c. Modify the original equations so that the path is traced in a clockwise direction.

d. Modify the original equations so that the path starts at $(0, 4)$. 

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LESSON 2 • SIMULATING LINEAR AND NONLINEAR MOTION 125
e. Modify the original equations so that the path is an ellipse crossing the
x-axis at ±8 and the y-axis at ±2.
f. Modify your equations in Part e so that the path is traced out clockwise
starting from (–8, 0).

Checkpoint

In this investigation, you examined how the motion of a point following a
counterclockwise circular path can be modeled with parametric equations.
The fundamental equations are

\[ x = A \cos Bt \]
\[ y = A \sin Bt, \]

where \( B \) is the angular velocity of the moving point

- **a** Describe the center and radius of this circular path.
- **b** What is the location of the point following this path when \( t = 0 \)?
- **c** How would you modify the equations so that the path is traced in a
clockwise direction?
- **d** How would you modify the equations so the center of the circular
motion is at the point with coordinates \((p, q)\)? Explain.
- **e** How would you modify the equations to produce an elliptical path with
center at the origin?

*Be prepared to share and explain your responses.*

On Your Own

A circular disk with center at \( P(0, 2) \) has a radius of 3. The disk rotates counter-
clockwise at \( 6\pi \) radians per second. Point \( Q \) on the disk has coordinates \((3, 2)\)
when \( t = 0 \).

- **a.** Sketch this situation.
- **b.** Write parametric equations modeling the motion of point \( Q \) for any time \( t \).
  Locate point \( Q \) when \( t = 0.1 \) and \( t = 0.5 \).
- **c.** What setting of \( \text{Tmax} \) will ensure that point \( Q \) makes exactly one revolution?
  Explain.
- **d.** Describe the settings for the parameter \( t \) that provide you with a good visual
  model of the path of point \( Q \).
1. In archery, as in other target shooting, the archer sets his sights so that when an arrow is shot at a particular distance, the arrow should hit the bull’s-eye.

   a. Suppose an arrow leaves a bow at about 150 feet per second. Taking into consideration the effect of gravity, estimate an angle at which you think the archer should shoot so that the arrow hits the bull’s-eye located 100 feet away. (Assume the center of the bull’s-eye is at the same height as the release point of the arrow.)

   b. Find equations for the $x$- and $y$-components of the shot at any time $t$ using your estimated angle.

   c. Check the adequacy of your model. How long does it take the arrow to reach the target? Does it hit the correct spot?

   d. If your model is not very accurate, modify the angle of aim until it gives better results. What angle seems to be the best?

2. Many people enjoy the game of darts. There are even national darts tournaments. To play darts, you stand 8 feet from a dart board and throw three darts per round, seeking to score points totaling 500.

   a. Suppose you are most accurate when you throw a dart at a 20° angle to the horizontal. If the bull’s-eye is at the same height as your release point, about what initial velocity must you impart to the dart to hit the bull’s-eye?

   b. How long is the thrown dart in the air?

   c. Suppose your dart-throwing opponent is most accurate when she throws at an initial velocity of 30 feet per second. At what angle should she throw her dart to hit the bull’s-eye?

   d. About how long is her dart in the air?
3. Baseball pitchers, such as Kerry Wood of the Chicago Cubs, can throw a fastball pitch at about 100 mph. Wood is about 6 ft, 4 in. tall and releases the ball about 5 ft above the ground. The pitcher’s mound is about 12 in. higher than the surrounding playing field. The pitching rubber itself is about 59 ft from the front of home plate.

a. Make a sketch of the situation in which Kerry Wood releases the ball 4 ft in front of the pitching rubber. Is his point of release 5 or 6 ft above the field?

b. At what angle to the horizontal should Wood throw the ball for it to cross the front of the plate 2 ft above the ground if gravity is the only other force acting on the ball?

c. Suppose that, due to the upward spin Wood puts on the ball, the effects of gravity are negated over the distance to the plate. Write parametric equations that describe the position of the ball \( t \) seconds after release, if the ball is thrown so that it will cross the front of home plate about 2 ft above the ground.

   - At what angle to the horizontal does Wood release the ball?
   - About how long does it take the pitch to get to the plate?

4. Joan Embery has spent much of her life researching the behavior of gorillas. Before examining injured gorillas, she must use a tranquilizer dart gun to sedate them. Her tranquilizer dart gun shoots darts at about 650 feet per second. Suppose Joan shot a dart (aimed horizontally at 5 ft above the ground) at a large injured gorilla 400 ft away.

a. Would the dart reach the gorilla? Explain.

b. How should Joan’s aim be adjusted so that she will hit the gorilla at a point somewhere between 2 and 5 ft above the ground? Test your best model with a graphing calculator or graphing software and sketch the result.

c. How much leeway does Joan have in choosing the angle at which to shoot?

5. In the diagram below, a pulley with 5 cm radius is centered at \( O(0, 0) \). A second pulley with 2 cm radius is centered at \( B(10, 0) \). The pulley at \( O \) is rotating counterclockwise at 2 revolutions per second.

a. Find the angular velocity of the pulley with center at point \( B \).

b. Write parametric equations that model the position of point \( A \) as the pulley rotates around its center \( O \) starting at \( (5, 0) \).

c. Model the position of point \( C \) with parametric equations, if its starting position is \( (12, 0) \).
d. Set your graphing calculator or computer graphing software on Simul (simultaneous), enter the algebraic descriptions of both circles, and display the graphs.

e. Set $T_{\text{min}}$ and $T_{\text{max}}$ so that one revolution around the circle with center $O$ is graphed. What happens to the graph of the related motion around the circle with center $B$? Explain.

f. Set $T_{\text{min}}$ and $T_{\text{max}}$ so that one revolution around the circle with center $B$ is graphed. What happens to the graph of the related motion around the circle with center $O$? Explain.

Organizing

1. The graph of $y = 3x + 2$ is a line.
   a. If $x = t$ is one of the parametric equations for this line, write the equation that expresses $y$ as a function of $t$.
   b. Display the graph of the parametric equations. Is the graph a line?
   c. Do the equation $y = 3x + 2$ and the set of parametric equations give the same line when graphed? If not, what would you change to make them look the same in the viewing window $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$?

2. Consider these two sets of parametric equations:
   \[
   \begin{aligned}
   x &= t \\
   y &= 2t - 3
   \end{aligned}
   \quad
   \begin{aligned}
   x &= 2t + 2 \\
   y &= 4t + 1
   \end{aligned}
   
   a. Draw the graph of each pair of parametric equations on its own coordinate system. How are the two graphs related?
   b. Find the $x$- and $y$-values for both sets of parametric equations when $t = 1$. Compare the results.
   c. For each pair of equations, find a value of $t$ that gives the point $(0, -3)$.

3. Consider the following two sets of parametric equations:
   \[
   \begin{aligned}
   x &= t \\
   y &= 2t
   \end{aligned}
   \quad
   \begin{aligned}
   x &= 3t \\
   y &= 6t
   \end{aligned}
   
   a. Compare the graphs of these pairs of equations.
   b. For each pair of equations, describe the rates at which $x$ and $y$ change with respect to $t$.
   c. For each pair of equations, describe the rate at which $t$ changes with respect to $x$ and with respect to $y$.
   d. How could you use the information in Parts b and c to determine the rate of change of $y$ with respect to $x$?
   e. Compare the rates at which a point moves along the graphs of the two pairs of parametric equations.
4. The graph of each pair of parametric equations below is a line.

   i. \( x = 2t \quad y = 3t \)
   ii. \( x = -3t \quad y = 2t \)
   iii. \( x = 2t + 1 \quad y = -t - 2 \)
   iv. \( x = at \quad y = bt \)

   a. For each pair of parametric equations, determine the slope of the line.
   b. For each pair of parametric equations, combine the two equations into a single equation that expresses \( y \) as a function of \( x \).

5. In the figure at the right, there are four circles with the same center, \( O \). The circles have radii of 1, 2, 3.5, and 5 units. Arc \( AB \) has length 1 unit.

   a. What is the radian measure of \( \angle O \)?
   b. A size transformation with center \( O \) will transform arc \( AB \) to each other arc. Find the magnitude of each such transformation.
   c. Find the lengths of arcs \( CD \), \( EF \), and \( GH \).
   d. How do similar circles and their arcs help make the definition of a radian more reasonable?

**Reflecting**

1. If you were an athlete who wished to throw, kick, or hit a ball as far as possible, would you rather do it on the Moon, Earth, or Jupiter? Explain your reasoning.

2. The study of parametric equations is enhanced by the use of the graphing and table-building capabilities of graphing calculators and computer graphing software. Identify the capability you find most helpful for each of the following tasks when modeling projectile motion such as that of a kicked soccer ball and explain your choice.

   a. Visualize the path.
   b. Determine the maximum height.
   c. Determine the horizontal distance traveled.
   d. Determine a point at which the vertical velocity is 0.
3. Linear motion through the origin can be modeled with parametric equations of the form

\[ x = At \cos \theta \]
\[ y = At \sin \theta \]

where \( A \) and \( \theta \) are constants.

a. What does \( A \) represent? What does \( \theta \) represent?
b. How would these equations be modified if \( (x, y) = (4, 0) \) when \( t = 0 \)?
c. How would these equations be modified if \( (x, y) = (0, -3) \) when \( t = 0 \)?

4. Why is approximately the same motion defined by each pair of parametric equations?

\[ x = 3t \cos 42^\circ \]
\[ y = 3t \sin 42^\circ \]
\[ x = 2.2294t \]
\[ y = 2.0074t \]

5. According to Morris Kline, a former Professor of Mathematics at New York University, “The advantage of radians over degrees is simply that it is a more convenient unit... The point involved here is no different from measuring a mile in yards instead of inches.” Why do you think Kline believes the radian is a “more convenient unit”? (Morris Kline, Mathematics for Liberal Arts, Addison Wesley, 1967. p. 423)

1. Parametric equations can be used to represent most familiar functions. Use a graphing calculator or computer graphing software to display the graph of each function and then make a sketch of the graph on your paper. Describe the shape of the graph. Then combine the equations into a single equation relating \( x \) and \( y \).

a. \( x = t + 1, \ y = t + 2 \)

b. \( x = 2 - 3t, \ y = t + 5 \)

c. \( x = t, \ y = \frac{3}{t} \)

d. \( x = 4 \cos t, \ y = 4 \sin t \)

e. \( x = 4t - 2, \ y = 8t^2 \)

2. Suppose Tiger Woods drives a golf ball about 285 yards, which includes 50 yards of roll after it hits the ground. His drive leaves the club at about a 27° angle.

a. What is the initial velocity of the ball when it leaves the club?
b. What is the maximum height that the ball reaches?
c. How long is the ball in the air?
3. Northwest Airlines has hubs at Detroit, Memphis, and Minneapolis. Flights from Memphis to Seattle and Detroit to Los Angeles leave at the same time and cruise at 32,000 feet. On a particular day, a 70 mph upper level northwest wind is blowing. The heading of the flight out of Memphis is 307° and the heading of the flight out of Detroit is 260°. The still air speed of each airliner is 600 mph.

a. Memphis is 400 miles west and 500 miles south of Detroit. Write parametric equations for the paths each plane will follow if they head directly toward their destinations.

b. On what heading should each plane steer to counteract the effects of the wind?

c. What is the wind-affected speed of each plane?

d. Do the paths of these two aircraft intersect? If so, what are the coordinates of the point of intersection? Is there danger of a mid-air collision? Explain your reasoning.

e. Seattle is about 1,800 mi from Memphis and Los Angeles is about 2,000 miles from Detroit. About how long will it take each plane to reach its destination at its wind-affected speed?

4. Two freighters leave port at the same time. On a coordinate system, one port is located at A(20, 0) and the other at B(−15, −4). The freighter Mystic Star leaves port A steaming west at 8 knots on a heading of 280°; the Queensland steams east from Port B on a heading of 70° at 10 knots.

a. Write parametric equations for each of these routes.

b. Represent the path of each of these freighters on your graphing calculator or computer graphing software. Sketch the paths on your paper.
c. Do the paths of the freighters cross? Is there a danger that the freighters will collide? Explain.

d. What is the location of the Mystic Star when the Queensland crosses its path? What is the location of the Queensland when the Mystic Star crosses its path?

5. In Investigation 4, you saw that in a coordinate system, a circular orbit with center at the origin can be described by parametric equations of the form:

\[ x = r \cos At \quad \text{and} \quad y = r \sin At. \]

Similarly, an elliptical orbit with center at the origin can be described by parametric equations of the form:

\[ x = a \cos At \quad \text{and} \quad y = b \sin At, \quad \text{where} \quad a \neq b. \]

In the case of a satellite or space station orbiting Earth, the path is an elliptical orbit, but its center is not the center of Earth. Earth’s center is one of two foci of the orbit, as indicated in the diagram below. The apogee is the point on the orbit farthest from Earth. The perigee is the point on the orbit closest to Earth.

a. In the case of a space station, NASA expects the altitude of apogee \( H_a \) to be 400 miles and the altitude of perigee \( H_p \) to be 200 miles. Using the diagram above (not drawn to scale) and assuming the radius of Earth is about 4,000 mi, develop parametric models of Earth and of the orbit of a space station around Earth.

b. Simulate the motion of the space station in orbit around Earth and verify the coordinates of the apogee and perigee.
1. If $\frac{8}{x-3} = \frac{5}{2}$, then $x =$
   
   (a) 14  (b) 0.2  (c) 19  (d) 6.2  (e) 6.1

2. \( \frac{3}{x} - \frac{4}{y} = \)
   
   (a) \( \frac{3y - 4x}{xy} \)  (b) \( \frac{3x - 4y}{xy} \)  (c) \( \frac{-1}{x - y} \)  (d) \( \frac{-1}{x + y} \)  (e) \( \frac{-1}{xy} \)

3. Find the value of $b$, if the slope of the line $12x - by = 15$ is $-36$.
   
   (a) \( \frac{5}{4} \)  (b) $-3$  (c) $3$  (d) $\frac{1}{3}$  (e) $-\frac{1}{3}$

4. If $f(x) = x^2 + 2x - 3$, then $f(a - 1) =$
   
   (a) $a^2 + 4a + 4$  (b) $a^2 - 5$  (c) $a^2 + 5$
   
   (d) $a^2 - 4$  (e) $a^2 + 2a - 4$

5. Solve $6m^2 - 11m = 35$ for $m$.
   
   (a) $m = \frac{5}{3}$ or $m = -\frac{7}{2}$  (b) $m = -\frac{5}{3}$ or $m = \frac{7}{2}$  (c) $m = -\frac{3}{5}$ or $m = \frac{2}{7}$
   
   (d) $m = \frac{3}{5}$ or $m = -\frac{2}{7}$  (e) no real solution
6. The solutions to \(2 |x - 5| - 3 = 15\) are \(x =\)

(a) \(-14\) and \(14\)  
(b) \(-4\) and \(4\)  
(c) \(14\)  
(d) \(-4\)  
(e) \(-4\) and \(14\)

7. If \((x + 4)^3 = 50\), then which of the following best approximates \(x\)?

(a) \(3.58\)  
(b) \(0.92\)  
(c) \(-0.32\)  
(d) \(3.78\)  
(e) \(2.32\)

8. \(\sin (-x) =\)

(a) \(\cos (-x)\)  
(b) \(-\sin x\)  
(c) \(\sin x\)  
(d) \(\cos x\)  
(e) \(\frac{1}{\sin x}\)

9. If \(3^x = 54\), then which of the following best approximates \(x\)?

(a) \(3.631\)  
(b) \(18\)  
(c) \(162\)  
(d) \(0.056\)  
(e) \(3.780\)

10. Solve \(\sqrt{2x + 6} = 6\).

(a) \(105\)  
(b) \(2\)  
(c) \(-2.32\)  
(d) \(216\)  
(e) none of these
In this unit, you investigated how vectors can be used to model navigation routes. In that setting, the idea of the sum or resultant vector was found to be useful in determining courses of ships and airplanes that were moving with a constant velocity. You also examined how vectors, their components, and the parametric equations derived from the components can be used to analyze linear and projectile motion. These ideas are also useful in representing circular motion of a point on a disc rotating at a constant velocity. Using ideas of transformations of graphs, you were able to develop models for elliptical orbits. In this final lesson, you will have the opportunity to review and apply these important ideas in new contexts.

1. The Gulf Stream is a warm ocean current flowing from the Gulf of Mexico along the east coast of the United States. It is about 50 nautical miles wide.
   a. Off New York City, the Gulf Stream flows at about 3 knots on a heading of about 35°. Sketch a vector representing the Gulf Stream current.
   b. Suppose the freighter Morocco out of New York City is steaming at 12 knots on a heading of 100° when it meets the Gulf Stream. Represent the Morocco's course with a vector.
   c. Make a vector model showing the effect of the Gulf Stream on the Morocco as it moves across the Gulf Stream.
   d. What course should the Morocco steer to cross the Gulf Stream and remain on its planned course?
   e. How long will it take the Morocco to cross the Gulf Stream?

2. Preliminary testing of robocarriers (as pictured) used in paper mills involves parallel tracks 85 meters long. Suppose a robocarrier programmed to travel 20 meters per minute (m/min) is placed on one track and a robocarrier programmed to travel at 30 m/min is placed on the second track.
   a. Model the motion of the two robocarriers under the conditions that the testing of the second robocarrier begins 30 seconds after the first.
b. Which robocarrier reaches the end of the test track first? How close to the end of its track is the other robocarrier when this happens?

c. Does the second robocarrier overtake the first? If so, at what time?

d. How could the rates of the robocarriers be adjusted so that both robocarriers arrive at the end of their tracks at about the same time?

3. In October 2001, golfers Tiger Woods and Annika Sorenstam beat David Duval and Karrie Webb in a primetime golf match. Annika Sorenstam, the top woman golfer in 2001, typically can hit her drives 260 yards, including about 30 yards of roll. Suppose her drives leave the tee at about a 25° angle.

a. Sketch a vector model for the position of the ball of a typical Sorenstam drive at time $t$ seconds.

b. Write rules for the components. Use variables for any unknown quantities.

c. What initial velocity must her club head impart to the ball for it to stay in the air for 200 yards?

d. How long is the ball in the air?

e. How far will Sorenstam hit her drive if she is hitting with a 5 mph wind?

f. How far will Sorenstam hit her drive if she hits from a tee region that is elevated 30 feet and there is no wind?

4. Laserdiscs in the constant angular velocity (CAV) format rotate at 1,800 revolutions per minute. Such a disc is shown at the right. Each side of a 12-inch CAV disc has 54,000 tracks; each track contains one frame of a motion picture.

a. Motion pictures are recorded at 30 frames per second. How many CAV laserdiscs are needed to record a 1-hour, 59-minute motion picture?

b. Express the angular velocity of a CAV laserdisc in radians per second and in degrees per second.

c. Write parametric equations to describe the location of a point, $P$, on a track 4.5 inches from the center $O$, if $\overrightarrow{OP}$ points along the $x$-axis when $t = 0$. 
5. Write parametric equations describing each of the following paths of an object moving at a constant velocity.

a. A vertical line through the point (5, -2)

b. A line through the points (2, 4) and (6, -2)

c. A circle of radius 5 centered at the origin, traced counterclockwise, starting at the point (0, 5) when $t = 0$

d. An ellipse centered at the origin, traced counterclockwise, crossing the x-axis at ±8 and the y-axis at ±12

e. How would you modify the parametric equations in Parts c and d if the paths were to be traced out clockwise?

6. The following problem was posed by Neal Koblitz in the March 1988 issue of the *American Mathematical Monthly*. The problem is one of several applied problems given to his calculus classes at the University of Washington. Solve the problem using methods developed in this unit.

You are standing on the ground at point B (see diagram), a distance of 75 ft from the bottom of a Ferris wheel with radius 20 ft. Your arm is at the same level as the bottom of the Ferris wheel. Your friend is on the Ferris wheel, which makes one revolution (counterclockwise) every 12 seconds. At the instant when she is at point A, you throw a ball to her at 60 ft/sec at an angle of 60° above the horizontal. Take $g = -32$ ft/sec², and neglect air resistance. Find the closest distance the ball gets to your friend. (Source: *American Mathematical Monthly*, March 1988, page 256.)
Checkpoint

Vectors and parametric equations are very useful for modeling and analyzing both linear and nonlinear motion.

a. What is a vector? What characteristics distinguish a vector from a number? From a segment? How are vectors added? Subtracted?

b. How many variables are involved in modeling motion with parametric equations? What is the significance of each variable?

c. Describe parametrically and graphically the following paths of a point P.
   - P is moving in the direction of $\theta$ at a constant velocity of 7 m/s.
   - P is propelled at an angle $\theta$ with an initial velocity of 100 m/s and then flies freely in the air.
   - P is on a circle with radius 2 m making 15 counterclockwise revolutions per minute.

d. How are parametric equations for circular motion similar to, and different from, those for elliptical motion? For linear motion? For projectile trajectories?

Be prepared to explain your responses and thinking to the entire class.

On Your Own

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.